Dynamic Random Walks on Clifford Algebras

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Abstract

Multiplicative random walks with dynamic transitions are defined on Clifford algebras of arbitrary signature. These multiplicative walks are then summed to induce additive walks on the algebra. Properties of both types of walks are considered, and limit theorems are developed.

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1 Introduction

Given nonnegative integers p and q, the Clifford algebra $\mathcal{C}\ell_{p,q}$ is a noncommutative associative algebra of dimension 2^{p+q}. Special cases include the complex numbers $\mathbb{C} \cong \mathcal{C}\ell_{0,1}$, the algebra of quaternions $\mathbb{H} \cong \mathcal{C}\ell_{0,2}$, the spacetime algebra $\mathcal{C}\ell_{1,3}$, the algebra of physical space $\mathcal{C}\ell_{3,0}$, and the n-particle fermion algebra $\mathcal{C}\ell_{n,n}$. Applications of Clifford algebras include electromagnetism, special relativity, quantum theory, and gravity.

More recent applications of Clifford algebras include image processing [5], automated geometric theorem proving [6], and computer vision [4]. In work related to computer vision, Perwass, Gebken, and Sommer [7] use Clifford algebras to discuss the estimation of points, lines, circles, etc. from uncertain data while keeping track of error propagation. Random walks are relevant in this context as models of error propagation.

In earlier work by the authors, Clifford methods were applied to the study of random graphs [10]. The second author has used Clifford methods to formulate random walks on the hypercube [9]. Time-homogeneous random walks on Clifford algebras have also been objects of recent study [8].

The current work follows the approach of the work on Heisenberg groups developed by Guillotin-Plantard and Schott [2]. Another work relating dynamical systems to Clifford algebras is Jadczyk's Clifford approach to quantum fractals [3].

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Definition 1.1. For fixed $n \geq 0$, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote the canonical orthonormal basis of $\mathbb{R}^n$. The 2^n-dimensional *Clifford algebra* of signature (p, q), where $p + q = n$, is defined as the associative algebra generated by the collection $\{\mathbf{e}_i\}$ along with the scalar $\mathbf{e}_0 = \mathbf{e}_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:

$$\mathbf{e}_i\, \mathbf{e}_j + \mathbf{e}_j\, \mathbf{e}_i = 0 \text{ for } i \neq j, \text{ and} \tag{1.1}$$

$$\mathbf{e}_i{}^2 = \begin{cases} 1, & \text{if } 1 \leq i \leq p \\ -1, & \text{if } p+1 \leq i \leq p+q = n. \end{cases} \tag{1.2}$$

The Clifford algebra of signature (p, q) is denoted $\mathcal{C}\ell_{p,q}$.

Generally the vectors generating the algebra do not have to be orthogonal. When they are orthogonal as in the definition above, the resulting multivectors are called *blades*.

Let $[n] = \{1, , 2, \ldots, n\}$ and let arbitrary subsets of $[n]$ be denoted by underlined Roman characters. The basis elements of $\mathcal{C}\ell_{p,q}$ can then be indexed by these finite subsets by writing $\mathbf{e}_{\underline{i}} = \prod_{k \in \underline{i}} \mathbf{e}_k$. Arbitrary elements of $\mathcal{C}\ell_{p,q}$ have the form

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}, \tag{1.3}$$

where $u_{\underline{i}} \in \mathbb{R}$ for each $\underline{i} \in 2^{[n]}$. For nonnegative integer k, the *degree-k* part of $u \in \mathcal{C}\ell_{p,q}$ will be defined by

$$\langle u \rangle_k = \sum_{|\underline{i}|=k} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}. \tag{1.4}$$

The *inner-product* of $u, v \in \mathcal{C}\ell_{p,q}$ is defined by

$$\langle u, v \rangle = \left\langle \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}\, , \sum_{\underline{i} \in 2^{[n]}} v_{\underline{i}}\, \mathbf{e}_{\underline{i}} \right\rangle = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\, v_{\underline{i}}\, . \tag{1.5}$$

Observe that for fixed multi-index $\underline{i} \in 2^{[n]}$ and arbitrary $u \in \mathcal{C}\ell_{p,q}$, $\langle u, \mathbf{e}_{\underline{i}} \rangle = u_{\underline{i}}$, the coefficient of the multivector $\mathbf{e}_{\underline{i}}$ in the canonical expansion of u. The inner product induces a *Clifford inner-product norm* by

$$\|u\|^2 = \langle u, u \rangle = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}{}^2. \tag{1.6}$$

2 Dynamic Walks on $\mathcal{C}\ell_{p,q}$

Given a random variable ξ, the *expectation* of ξ will be denoted by either $\langle \xi \rangle$ or $\mathbb{E}(\xi)$. Given a sequence of random variables $\{\xi_N\}$, the notation $\xi_N \xrightarrow{\mathcal{D}} \psi$ denotes *convergence in distribution* to the random variable ψ. The notation

$\xi_N \xrightarrow{\mathcal{P}} u$ denotes *convergence in probability* to u. The notation $\mathcal{U}(X)$ will denote the *uniform distribution* on the set X.

Fix nonnegative integers p and q, and let $n = p + q$. Following the approach of Guillotin-Plantard and Schott [2], let $\Sigma = (E, \mathcal{A}, \mu, T)$ be a dynamical system where $(E, \mathcal{A}, \mu)$ is a probability space and T is a transformation on E. Let $n \geq 1$ and $f_1, \ldots, f_n$ be functions defined on E with values in $[0, \frac{1}{n}]$.

Let $x \in E$ and let $\{\mathbf{e}_j\}_{1\leq j\leq n}$ be the unit coordinate vectors of $\mathbb{R}^n$. For every $i \geq 1$, the law of the random vector $M_i = ({x_1}^{(i)}, {x_2}^{(i)}, \ldots, {x_n}^{(i)})$ is given by

$$\mathbb{P}\left(M_i = z\right) = \begin{cases} f_j(T^i x) & \text{if } z = \mathbf{e}_j \\ \frac{1}{n} - f_j(T^i x) & \text{if } z = -\mathbf{e}_j \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In $\mathcal{C}\ell_{p,q}$ the multiplication satisfies

$$\mathbf{e}_i{}^2 = \begin{cases} 1 & \text{if } 0 \leq i \leq p \\ -1 & \text{if } p+1 \leq i \leq n, \end{cases} \tag{2.2}$$

and

$$\mathbf{e}_i\,\mathbf{e}_j = -\mathbf{e}_j\,\mathbf{e}_i \text{ if } 1 \leq i \neq j \leq n. \tag{2.3}$$

We are interested in the right dynamic random walk

$$\varsigma_k = ({x_1}^{(1)}, \ldots, {x_n}^{(1)}) \cdot ({x_1}^{(2)}, \ldots, {x_n}^{(2)}) \cdots ({x_1}^{(k)}, \ldots, {x_n}^{(k)}),\ k \geq 1. \tag{2.4}$$

The multiplicative walk (ς_k) can be visualized as a walk on a 2^n-vertex simple graph, as seen in Figure 1 for the walk in $\mathcal{C}\ell_{0,2}$. In this walk, there is no chance of visiting the same vertex in two consecutive steps. That is,

$$\mathbb{P}\left(\varsigma_k = \pm\mathbf{e}_{\underline{i}} \,|\, \varsigma_{k-1} = \pm\mathbf{e}_{\underline{i}}\right) = 0. \tag{2.5}$$

By definition, the walk (ς_k) alternates between blades of even and odd degree in $\mathcal{C}\ell_{p,q}$. That is, when $k \not\cong |\underline{i}| \pmod 2$

$$\mathbb{P}(\varsigma_k = \pm\mathbf{e}_{\underline{i}}) = 0. \tag{2.6}$$

Moreover, for arbitrary multi-index $\underline{i}$, the product of vectors indexed by elements of $\underline{i}$ each occurring with odd multiplicity taken with products of vectors indexed by elements outside $\underline{i}$ occurring with even multiplicities always results in blades of the form $\pm\mathbf{e}_{\underline{i}}$. This is stated formally in the following lemma.

Lemma 2.1. *When $k \geq |\underline{i}|$ and $k \equiv |\underline{i}| \pmod 2$,*

$$\mathbb{P}(\varsigma_k = \pm\mathbf{e}_{\underline{i}}) = \frac{1}{n^k} \sum_{\substack{\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd if } j\in\underline{i},\ \ell_j \text{ even if } j\notin\underline{i}}} \binom{k}{\ell_1, \ldots, \ell_n}. \tag{2.7}$$

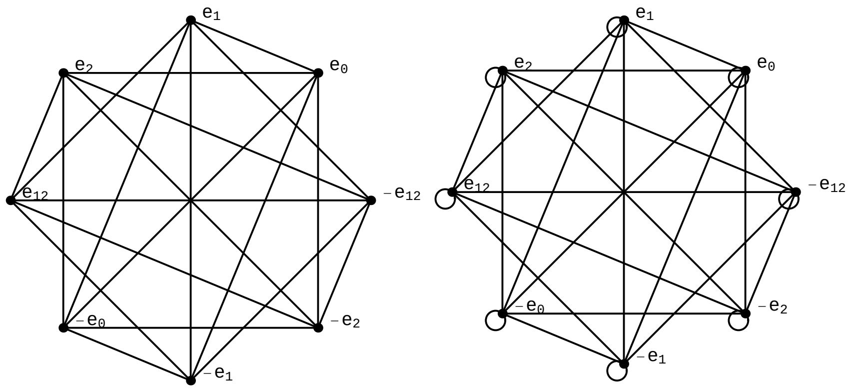

Figure 1: Graphs associated with walks (ς_k) and (τ_k) on $\mathcal{C}\ell_{0,2}$.

Another multiplicative walk is also defined. Let $g_0, g_1, \ldots, g_n$ be functions defined on E with values in $[0, \frac{1}{n+1}]$.

Let $x \in E$ and let $(\mathbf{e}_j)_{1 \leq j \leq n}$ be the unit coordinate vectors of $\mathbb{Z}^n$. Also define the unit scalar $\mathbf{e}_0 = \mathbf{e}_\emptyset = 1$. For every $i \geq 1$, the law of the random vector $Q_i = ({x_0}^{(i)}, {x_1}^{(i)}, {x_2}^{(i)}, \ldots, {x_n}^{(i)})$ is given by

$$\mathbb{P}\left(Q_i = z\right) = \begin{cases} g_j(T^i x) & \text{if } z = \mathbf{e}_j \\ \frac{1}{n+1} - g_j(T^i x) & \text{if } z = -\mathbf{e}_j \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

In addition to the multiplicative properties of (2.2) and (2.3), the multiplication satisfies the following for $0 \leq j \leq n$:

$$\mathbf{e}_0\, \mathbf{e}_j = \mathbf{e}_j\, \mathbf{e}_0 = \mathbf{e}_j. \tag{2.9}$$

The right dynamic random walk of interest now is

$$\tau_k = ({x_0}^{(1)}, {x_1}^{(1)}, \ldots, {x_n}^{(1)}) \cdots ({x_0}^{(n)}, {x_1}^{(k)}, \ldots, {x_n}^{(k)}),\ k \geq 1. \tag{2.10}$$

The multiplicative walk (τ_k) can be visualized as a walk on a 2^n-vertex graph having loops at all vertices, as seen in Figure 1 for the walk in $\mathcal{C}\ell_{0,2}$. Unlike the walk (ς_k), (τ_k) does not alternate between blades of even and odd degree in $\mathcal{C}\ell_{p,q}$.

Lemma 2.2. *Let $\mathbf{e}_{\underline{i}}$ be an arbitrary blade in $\mathcal{C}\ell_{p,q}$, and let k be an arbitrary positive integer such that $k \geq |\underline{i}|$. Then,*

$$\mathbb{P}\left(\tau_k = \pm\mathbf{e}_{\underline{i}}\right) = \frac{1}{(n+1)^k} \sum_{\ell_0=0}^{k} \binom{k}{\ell_0} \sum_{\substack{\ell_1+\cdots+\ell_n=k-\ell_0 \\ \ell_j \text{ odd if } j\in\underline{i},\, \ell_j \text{ even if } j\notin\underline{i}}} \binom{k-\ell_0}{\ell_1, \ldots, \ell_n}. \tag{2.11}$$

Proof. For arbitrary multi-index $\underline{i}$, the product of vectors indexed by elements of $\underline{i}$ each occurring with odd multiplicity taken with products of vectors indexed by elements outside $\underline{i}$ occurring with even multiplicities always results in blades of the form $\pm\mathbf{e}_{\underline{i}}$. The unit scalar may appear with any multiplicity. Recalling that τ_k is a product of k vectors, the following holds when $k \geq |\underline{i}|$:

$$\begin{aligned}
\mathbb{P}\left(\tau_k = \pm\mathbf{e}_{\underline{i}}\right) &= \frac{1}{(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k}{\ell_0,\ldots,\ell_n} \\
&= \frac{1}{(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \frac{k!}{\ell_0!\cdots\ell_n!} \\
&= \frac{1}{(n+1)^k} \sum_{\ell_0=0}^{k} \frac{1}{\ell_0!} \sum_{\substack{\ell_1+\cdots+\ell_n=k-\ell_0 \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \frac{k!}{(k-\ell_0)!} \frac{(k-\ell_0)!}{\ell_1!\cdots\ell_n!} \\
&= \frac{1}{(n+1)^k} \sum_{\ell_0=0}^{k} \frac{k!}{\ell_0!(k-\ell_0)!} \sum_{\substack{\ell_1+\cdots+\ell_n=k-\ell_0 \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k-\ell_0}{\ell_1,\ldots,\ell_n}.
\end{aligned} \tag{2.12}$$

□

2.1 Expectation

The expectation of an arbitrary step of the right dynamic walk (ς_k) can be computed directly.

Lemma 2.3. *Let $(\varsigma_k)_{k\geq1}$ be the Clifford-valued random walk defined by* (2.4)*. Then,*

$$\langle \varsigma_k \rangle = \prod_{i=1}^{k} \left(\sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j \right). \tag{2.13}$$

Proof. From (2.1), one sees

$$\langle M_i \rangle = \sum_{j=1}^{n} \left(f_j(T^i x) - \left(\frac{1}{n} - f_j(T^i x) \right) \right) \mathbf{e}_j = \sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j. \tag{2.14}$$

Using independence of the random vectors (M_i) and the definition of the random walk (ς_k), the expectation of ς_k can then be computed directly.

$$\langle \varsigma_k \rangle = \left\langle \prod_{i=1}^{k} M_i \right\rangle = \prod_{i=1}^{k} \langle M_i \rangle = \prod_{i=1}^{k} \left(\sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j \right). \tag{2.15}$$

□

Define the product signature function $\varpi : 2^{[n]} \times [n] \to \{0,1\}$ by

$$\varpi(\underline{i}, \{j\}) = (|\{\iota \in \underline{i} : \iota > j\}| + |\underline{i} \cap \{j\} \cap \{p+1, \dots, n\}|) \pmod 2. \tag{2.16}$$

In $\mathcal{C}\ell_{p,q}$, the product signature function satisfies $\mathbf{e}_{\underline{i}}\mathbf{e}_j = (-1)^{\varpi(\underline{i},\{j\})}\mathbf{e}_{\underline{i}\triangle\{j\}}$, where $\triangle$ represents the set symmetric difference.

Proposition 2.4. *Define the notation $s_{\underline{i}}^{\pm}(k) = \mathbb{P}(\varsigma_k = \pm\mathbf{e}_{\underline{i}})$. The probability density functions $s_{\underline{i}}^{\pm}(k)$ satisfy the following recurrence relation:*

$$s_{\underline{i}}^{+}(k) = \begin{cases} 0 & \text{if } k < |\underline{i}|, \\ f_j(Tx) & \text{if } \underline{i} = \{j\} \text{ and } k = |\underline{i}| = 1. \end{cases} \tag{2.17}$$

$$s_{\underline{i}}^{-}(k) = \begin{cases} 0 & \text{if } k < |\underline{i}|, \\ \frac{1}{n} - f_j(Tx) & \text{if } \underline{i} = \{j\} \text{ and } k = |\underline{i}| = 1. \end{cases} \tag{2.18}$$

When $k > |\underline{i}|$ or $k = |\underline{i}| > 1$,

$$\begin{aligned} s_{\underline{i}}^{+}(k) = \sum_{j=1}^{n} &\Big((1 - \varpi(\underline{i}, \{j\}))f_j(T^k x)s_{\underline{i}\triangle\{j\}}^{+}(k-1)\Big) \\ &+ \sum_{j=1}^{n} \left(\varpi(\underline{i}, \{j\}) \left(\frac{1}{n} - f_j(T^k x)\right) s_{\underline{i}\triangle\{j\}}^{-}(k-1)\right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} s_{\underline{i}}^{-}(k) = \sum_{j=1}^{n} &\Big((1 - \varpi(\underline{i}, \{j\}))f_j(T^k x)s_{\underline{i}\triangle\{j\}}^{-}(k-1)\Big) \\ &+ \sum_{j=1}^{n} \left(\varpi(\underline{i}, \{j\}) \left(\frac{1}{n} - f_j(T^k x)\right) s_{\underline{i}\triangle\{j\}}^{+}(k-1)\right) \end{aligned} \tag{2.20}$$

Proof. The conditions $s_{\underline{i}}^{\pm}(k) = 0$ when $k < |\underline{i}|$, $s_{\{j\}}^{+}(1) = f_j(Tx)$, and $s_{\{j\}}^{-}(1) = \frac{1}{n} - f_j(Tx)$ are clear from the definition of the walk (ς_k).

Let $\sqcup$ denote disjoint union. When $k > |\underline{i}|$ or $k = |\underline{i}| > 1$, the definition of (ς_k) dictates that $\varsigma_k = \mathbf{e}_{\underline{i}}$ if and only if one of the following eight cases occurs for some $j \in \{1, \dots, n\}$:

- $\varsigma_{k-1} = \pm\mathbf{e}_{\underline{i}\setminus\{j\}}$, $M_k = \pm\mathbf{e}_j$, and $\varsigma_{k-1}M_k = \mathbf{e}_{\underline{i}}$.
- $\varsigma_{k-1} = \pm\mathbf{e}_{\underline{i}\sqcup\{j\}}$, $M_k = \pm\mathbf{e}_j$, and $\varsigma_{k-1}M_k = \mathbf{e}_{\underline{i}}$.

Similar conditions hold for $\varsigma_k = -\mathbf{e}_{\underline{i}}$. The probabilities associated with these conditions are exactly the values appearing in the recurrence of the proposition. □

Straightforward calculation reveals the expectation of the k^{th} step of the random walk.

$$\langle \varsigma_k \rangle = \sum_{\underline{i} \in 2^{[n]}} \left(s^+_{\underline{i}}(k) - s^-_{\underline{i}}(k) \right) \mathbf{e}_{\underline{i}}. \tag{2.21}$$

Proposition 2.5. *For any $k > 0$, the expectation of the k^{th} step of the random walk satisfies the following:*

$$\langle \varsigma_1 \rangle = \sum_{j=1}^{n} \left(2 f_j(Tx) - \frac{1}{n} \right) \mathbf{e}_j, \tag{2.22}$$

and when $k > |\underline{i}|$ or $k = |\underline{i}| > 1$,

$$\langle \varsigma_k \rangle = \sum_{\underline{i} \in 2^{[n]}} \sum_{j=1}^{n} \left(f_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n} \right) \left(s^+_{\underline{i} \triangle \{j\}}(k-1) - s^-_{\underline{i} \triangle \{j\}}(k-1) \right) \mathbf{e}_{\underline{i}}. \tag{2.23}$$

In particular,

$$\mathbb{E}\left(\langle \varsigma_k, \mathbf{e}_{\underline{i}} \rangle \right) = \sum_{j=1}^{n} \left(f_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n} \right) \mathbb{E}\left(\langle \varsigma_{k-1}, \mathbf{e}_{\underline{i} \triangle \{j\}} \rangle \right). \tag{2.24}$$

Proof. Proof is by direct calculation using Proposition 2.4. Beginning with the observation (2.21),

$$\begin{aligned}
\mathbb{E}\left(\langle \varsigma_k, \mathbf{e}_{\underline{i}} \rangle \right) &= s^+_{\underline{i}}(k) - s^-_{\underline{i}}(k) \\
&= \sum_{j=1}^{n} \left((1 - \varpi(\underline{i}, \{j\})) f_j(T^k x) s^+_{\underline{i} \triangle \{j\}}(k-1) \right) \\
&\quad + \sum_{j=1}^{n} \left(\varpi(\underline{i}, \{j\}) \left(\frac{1}{n} - f_j(T^k x) \right) s^-_{\underline{i} \triangle \{j\}}(k-1) \right) \\
&\quad - \sum_{j=1}^{n} \left((1 - \varpi(\underline{i}, \{j\})) f_j(T^k x) s^-_{\underline{i} \triangle \{j\}}(k-1) \right) \\
&\quad - \sum_{j=1}^{n} \left(\varpi(\underline{i}, \{j\}) \left(\frac{1}{n} - f_j(T^k x) \right) s^+_{\underline{i} \triangle \{j\}}(k-1) \right) \\
&= \sum_{j=1}^{n} \left(s^+_{\underline{i} \triangle \{j\}}(k-1) - s^-_{\underline{i} \triangle \{j\}}(k-1) \right) \left(f_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n} \right) \\
&= \sum_{j=1}^{n} \left(f_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n} \right) \mathbb{E}\left(\langle \varsigma_{k-1}, \mathbf{e}_{\underline{i} \triangle \{j\}} \rangle \right).
\end{aligned} \tag{2.25}$$

Linearity of expectation then gives

$$
\begin{aligned}
\langle \varsigma_k \rangle &= \sum_{\underline{i} \in 2^{[n]}} \mathbb{E}\left(\left\langle \varsigma_k, \mathbf{e}_{\underline{i}} \right\rangle\right) \mathbf{e}_{\underline{i}} \\
&= \sum_{\underline{i} \in 2^{[n]}} \sum_{j=1}^{n} \left(f_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n} \right) \mathbb{E}\left(\left\langle \varsigma_{k-1}, \mathbf{e}_{\underline{i} \triangle \{j\}} \right\rangle\right) \mathbf{e}_{\underline{i}} \\
&= \sum_{\underline{i} \in 2^{[n]}} \sum_{j=1}^{n} \left(f_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n} \right) \left(s^{+}_{\underline{i} \triangle \{j\}}(k-1) - s^{-}_{\underline{i} \triangle \{j\}}(k-1) \right) \mathbf{e}_{\underline{i}}. \qquad (2.26)
\end{aligned}
$$

□

Corollary 2.6. *For any $k > 0$ and $\underline{i} \in 2^{[n]}$, the expectation of $\langle \varsigma_k, \mathbf{e}_{\underline{i}} \rangle$ is given by*

$$
\begin{aligned}
\mathbb{E}\left(\left\langle \varsigma_k, \mathbf{e}_{\underline{i}} \right\rangle\right) &= \sum_{j_0=1}^{n} \cdots \sum_{j_{k-1}=1}^{n} \left(f_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n} \right) \\
&\quad \times \left[\prod_{\ell=1}^{k-1} \left(f_{j_\ell}(T^{k-\ell} x) - \frac{1}{n} \varpi\left(\left(\underline{i} \mathop{\triangle}_{1 \le m \le \ell-1} \{j_m\} \right), \{j_\ell\} \right) \right) \right]. \qquad (2.27)
\end{aligned}
$$

Proof. The result follows from Proposition 2.5 and back-substitution. □

As in Lemma 2.3, the expectation of an arbitrary step of the right dynamic walk (τ_k) can also be computed directly.

Lemma 2.7. *Let $(\tau_k)_{k \ge 1}$ be the Clifford-valued random walk defined by* (2.10)*. Then,*

$$
\langle \tau_k \rangle = \prod_{i=1}^{k} \left(\sum_{j=0}^{n} \left(2 g_j(T^i x) - \frac{1}{n+1} \right) \mathbf{e}_j \right). \qquad (2.28)
$$

Proof. From (2.8), one sees

$$
\langle Q_i \rangle = \sum_{j=0}^{n} \left(g_j(T^i x) - \left(\frac{1}{n+1} - g_j(T^i x) \right) \right) \mathbf{e}_j = \sum_{j=0}^{n} \left(2 g_j(T^i x) - \frac{1}{n+1} \right) \mathbf{e}_j. \qquad (2.29)
$$

Using independence of the random vectors (Q_i) and the definition of the random walk (τ_k), the expectation of τ_k can then be computed directly.

$$
\langle \tau_k \rangle = \left\langle \prod_{i=1}^{k} Q_i \right\rangle = \prod_{i=1}^{k} \langle Q_i \rangle = \prod_{i=1}^{k} \left(\sum_{j=0}^{n} \left(2 g_j(T^i x) - \frac{1}{n+1} \right) \mathbf{e}_j \right). \qquad (2.30)
$$

□

The expected values of the components of τ_k can be computed recursively.

Proposition 2.8. *Define the notation* $t^{\pm}_{\underline{i}}(k) = \mathbb{P}(\tau_k = \pm\mathbf{e}_{\underline{i}})$. *The probability density functions* $t^{\pm}_{\underline{i}}(k)$ *satisfy the following recurrence relation:*

$$t^{+}_{\underline{i}}(k) = \begin{cases} 0 & \text{if } k < |\underline{i}|, \\ g_j(Tx) & \text{if } \underline{i} = \{j\} \text{ and } k = |\underline{i}| = 1. \end{cases} \tag{2.31}$$

$$t^{-}_{\underline{i}}(k) = \begin{cases} 0 & \text{if } k < |\underline{i}|, \\ \frac{1}{n+1} - g_j(Tx) & \text{if } \underline{i} = \{j\} \text{ and } k = |\underline{i}| = 1. \end{cases} \tag{2.32}$$

When $k > |\underline{i}|$ *or* $k = |\underline{i}| > 1$,

$$\begin{aligned} t^{+}_{\underline{i}}(k) = t^{+}_{\underline{i}}(k-1)g_0(T^k x) &+ \left(\frac{1}{n+1} - g_0(T^k x)\right) t^{-}_{\underline{i}}(k-1) \\ &+ \sum_{j=1}^{n} \Big((1 - \varpi(\underline{i}, \{j\}))g_j(T^k x)t^{+}_{\underline{i}\triangle\{j\}}(k-1)\Big) \\ &+ \sum_{j=1}^{n} \left(\varpi(\underline{i}, \{j\}) \left(\frac{1}{n+1} - g_j(T^k x)\right) t^{-}_{\underline{i}\triangle\{j\}}(k-1)\right) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} t^{-}_{\underline{i}}(k) = t^{-}_{\underline{i}}(k-1)g_0(T^k x) &+ \left(\frac{1}{n+1} - g_0(T^k x)\right) t^{+}_{\underline{i}}(k-1) \\ &+ \sum_{j=1}^{n} \Big((1 - \varpi(\underline{i}, \{j\}))g_j(T^k x)t^{-}_{\underline{i}\triangle\{j\}}(k-1)\Big) \\ &+ \sum_{j=1}^{n} \left(\varpi(\underline{i}, \{j\}) \left(\frac{1}{n+1} - g_j(T^k x)\right) t^{+}_{\underline{i}\triangle\{j\}}(k-1)\right) \end{aligned} \tag{2.34}$$

Proof. The conditions $t^{\pm}_{\underline{i}}(k) = 0$ when $k < |\underline{i}|$, $t^{+}_{\{j\}}(1) = g_j(Tx)$, and $t^{-}_{\{j\}}(1) = \frac{1}{n+1} - g_j(Tx)$ are clear from the definition of the walk (τ_k).

Let $\sqcup$ denote disjoint union. When $k > |\underline{i}|$ or $k = |\underline{i}| > 1$, the definition of (τ_k) dictates that $\tau_k = \mathbf{e}_{\underline{i}}$ if and only if one of the following cases occurs for some $j \in \{0, \ldots, n\}$:

- $\tau_{k-1} = \pm\mathbf{e}_{\underline{i}}$, $Q_k = \pm\mathbf{e}_0$, and $\tau_{k-1}Q_k = \mathbf{e}_{\underline{i}}$.
- $\tau_{k-1} = \pm\mathbf{e}_{\underline{i}\setminus\{j\}}$, $Q_k = \pm\mathbf{e}_j$, and $\tau_{k-1}Q_k = \mathbf{e}_{\underline{i}}$.
- $\tau_{k-1} = \pm\mathbf{e}_{\underline{i}\sqcup\{j\}}$, $Q_k = \pm\mathbf{e}_j$, and $\tau_{k-1}Q_k = \mathbf{e}_{\underline{i}}$.

Similar conditions hold for $\tau_k = -\mathbf{e}_{\underline{i}}$. The probabilities associated with these conditions are exactly the values appearing in the recurrence of the proposition. $\square$

For convenience, define $\varpi(\underline{i}, \{j\}) = 0$ when $j = 0$.

Corollary 2.9. *The quantity* $t_{\underline{i}}(k) = t^{+}_{\underline{i}}(k) - t^{-}_{\underline{i}}(k)$ *satisfies the following recurrence:*

$$t_{\underline{i}}(k) = \begin{cases} 0 & \text{if } k < |\underline{i}|, \\ 2g_j(Tx) - \frac{1}{n+1} & \text{if } \underline{i} = \{j\} \text{ and } k = 1, \\ 2g_0(Tx) - \frac{1}{n+1} & \text{if } \underline{i} = \emptyset \text{ and } k = 1, \end{cases} \tag{2.35}$$

and when $k > |\underline{i}|$ *or* $k = |\underline{i}| > 1$,

$$t_{\underline{i}}(k) = \left(2g_0(T^k x) + \sum_{j=1}^{n} \left(g_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n+1} \right) \right) t_{\underline{i}}(k-1). \tag{2.36}$$

Proof. Proof is by direct computation using Proposition 2.8. $\square$

A recurrence relation for the expectation found in Lemma 2.7 is now revealed.

Corollary 2.10. *The expected value of the* k^{th} *step of the random walk* τ_k *satisfies the following recurrence:*

$$\langle \tau_1 \rangle = \sum_{j=0}^{n} \left(2g_j(Tx) - \frac{1}{n+1} \right) \mathbf{e}_j, \tag{2.37}$$

and when $k > |\underline{i}|$ *or* $k = |\underline{i}| > 1$,

$$\langle \tau_k \rangle \quad = \sum_{\underline{i} \in 2^{[n]}} \left[t_{\underline{i}}(k-1) \left(2g_0(T^k x) + \sum_{j=1}^{n} \left(g_j(T^k x) - \frac{\varpi(\underline{i}, \{j\})}{n+1} \right) \right) \mathbf{e}_{\underline{i}} \right]. \tag{2.38}$$

Proof. The stated result is a consequence of

$$\langle \tau_k \rangle = \sum_{\underline{i} \in 2^{[n]}} t_{\underline{i}}(k) \mathbf{e}_{\underline{i}}. \tag{2.39}$$

$\square$

Corollary 2.11. *For* $\underline{i} \in 2^{[n]}$ *and* $k > 0$, *the expected value of* $\langle \tau_k, \mathbf{e}_{\underline{i}} \rangle$ *is given by*

$$\begin{aligned} \mathbb{E}\left(\langle \tau_k, \mathbf{e}_{\underline{i}} \rangle \right) &= \sum_{j_0=0}^{n} \cdots \sum_{j_{k-1}=0}^{n} \left(g_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n+1} \right) \\ &\times \left[\prod_{\ell=1}^{k-1} \left(g_{j_\ell}(T^{k-\ell} x) - \frac{1}{n+1} \varpi\left(\left(\underline{i} \mathop{\triangle}_{1 \le m \le \ell-1} \{j_m\} \right), \{j_\ell\} \right) \right) \right]. \end{aligned} \tag{2.40}$$

Proof. The result follows from Corollary 2.10 and back-substitution. $\square$

The following lemma is a consequence of the multiplication in $\mathcal{C}\ell_{p,q}$ and will make computations more straightforward.

Lemma 2.12. *Let nonnegative integers p and q be given, and let $k > 1$ be an integer. Then, in $\mathcal{C}\ell_{p,q}$*

$$\left(\sum_{i=1}^{n} a_i\,\mathbf{e}_i\right)^k = \begin{cases} \left(\displaystyle\sum_{i=1}^{p} {a_i}^2 - \sum_{j=p+1}^{n} {a_j}^2\right)^{k/2} & \text{if } k \equiv 0 \pmod 2 \\ \left(\displaystyle\sum_{i=1}^{p} {a_i}^2 - \sum_{j=p+1}^{n} {a_j}^2\right)^{(k-1)/2} \displaystyle\sum_{i=1}^{n} a_i\,\mathbf{e}_i & \text{if } k \equiv 1 \pmod 2. \end{cases} \tag{2.41}$$

Remark 2.13. Throughout the remainder of the paper, notation will be simplified for the case $p = q$ by adopting the convention $(p - q)^0 = (2p - n)^0 \equiv 1$.

Letting $a_i = 1$ for $1 \le i \le n$ in Lemma 2.12 gives the following corollary.

Corollary 2.14. *Define $\gamma = \mathbf{e}_1 + \cdots + \mathbf{e}_n \in \mathcal{C}\ell_{p,q}$. Then,*

$$\gamma^{2k} = (2p - n)^k\,, \text{ and} \tag{2.42}$$

$$\gamma^{2k+1} = (2p - n)^k\,\gamma. \tag{2.43}$$

Corollary 2.15. *Let γ be defined as in Corollary 2.14. In $\mathcal{C}\ell_{p,q}$,*

$$(1 + \mathbf{e}_1 + \cdots + \mathbf{e}_n)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} (2p - n)^{\lfloor \ell/2 \rfloor} \gamma^{\ell \pmod 2}. \tag{2.44}$$

Proof. With $\gamma = \mathbf{e}_1 + \cdots + \mathbf{e}_n$, applying the binomial theorem gives

$$(1 + \gamma)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \gamma^\ell. \tag{2.45}$$

By Corollary 2.14, $\gamma^\ell = (2p - n)^{\lfloor \ell/2 \rfloor} \gamma^{\ell \pmod 2}$. $\square$

3 Limit theorems

Writing $\gamma = \mathbf{e}_1 + \cdots + \mathbf{e}_n$,

$$\begin{aligned}
\langle \tau_k \rangle &= \prod_{i=1}^{k} \left(\sum_{j=0}^{n} \left(2g_j(T^i x) - \frac{1}{n+1} \right) \mathbf{e}_j \right) \\
&= 2^k \sum_{0 \le j_1, \ldots, j_k \le n} \pm g_{j_1}(Tx) g_{j_2}(T^2 x) \cdots g_{j_k}(T^k x) \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_k} \\
&\quad + \sum_{\ell=1}^{k-1} \frac{1}{(n+1)^{k-\ell}} \sum_{m=0}^{k-\ell} \binom{k-\ell}{m} (2p-n)^{\lfloor m/2 \rfloor} \gamma^m \pmod 2 \\
&\quad \times 2^\ell \sum_{\substack{0 \le j_1, \ldots, j_\ell \le n \\ h_1 \ne \cdots \ne h_\ell \in [k]}} \pm g_{j_1}(T^{h_1} x) \cdots g_{j_\ell}(T^{h_\ell} x) \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_\ell} \\
&\qquad + \frac{(-1)^k}{(n+1)^k} \sum_{\ell=0}^{k} \binom{k}{\ell} (2p-n)^{\lfloor \ell/2 \rfloor} \gamma^\ell \pmod 2. \qquad (3.1)
\end{aligned}$$

Lemma 3.1. *As $k \to \infty$,*

$$\frac{1}{(n+1)^k} \sum_{\ell=0}^{k} \binom{k}{\ell} (2p-n)^{\lfloor \ell/2 \rfloor} \gamma^\ell \pmod 2 \to 0. \qquad (3.2)$$

Proof. Begin by writing

$$\begin{aligned}
\eta &= \frac{1}{(n+1)^k} \sum_{\ell=0}^{k} \binom{k}{\ell} (2p-n)^{\lfloor \ell/2 \rfloor} \gamma^\ell \pmod 2 \\
&= \frac{1}{(n+1)^k} \left(\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} \binom{k}{\ell} (2p-n)^{\ell/2} + \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{k} \binom{k}{\ell} (2p-n)^{(\ell-1)/2} \gamma \right). \qquad (3.3)
\end{aligned}$$

Let the polynomials ${\varphi_k}^+(z)$ and ${\varphi_k}^-(z)$ be defined by

$${\varphi_k}^+(z) = \frac{1}{(n+1)^k} \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{k} \binom{k}{\ell} z^{\ell/2}, \qquad (3.4)$$

$$\varphi_k^-(z) = \frac{1}{(n+1)^k} \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{k} \binom{k}{\ell} z^{(\ell-1)/2}, \qquad (3.5)$$

so that ${\varphi_k}^+(z^2) + z{\varphi_k}^-(z^2) = \left(\frac{z+1}{n+1} \right)^k$.

Letting $z = (2p-n)^{1/2}$ so that ${\varphi_k}^+(z^2) + \gamma {\varphi_k}^-(z^2) = \eta$, it becomes clear that $\lim\limits_{k \to \infty} \eta = 0$. $\square$

It now follows that $\lim\limits_{k\to\infty}\langle\tau_k\rangle$, if it exists, is given by

$$\begin{aligned}\lim_{k\to\infty}\langle\tau_k\rangle &= \lim_{k\to\infty} 2^k \sum_{0\le j_1,\ldots,j_k\le n} \pm g_{j_1}(Tx)g_{j_2}(T^2x)\cdots g_{j_k}(T^kx)\mathbf{e}_{j_1}\cdots\mathbf{e}_{j_k} \\ &\quad + \sum_{\ell=1}^{k-1}\frac{1}{(n+1)^{k-\ell}}\sum_{m=0}^{k-\ell}\binom{k-\ell}{m}(2p-n)^{\lfloor m/2\rfloor}\gamma^{m \pmod 2} \\ &\qquad \times 2^\ell \sum_{\substack{0\le j_1,\ldots,j_\ell\le n\\ h_1\neq\cdots\neq h_\ell\in[k]}} \pm g_{j_1}(T^{h_1}x)\cdots g_{j_\ell}(T^{h_\ell}x)\mathbf{e}_{j_1}\cdots\mathbf{e}_{j_\ell} \end{aligned} \tag{3.6}$$

Remark 3.2. In the homogeneous random walk with $g_k(x)\equiv\frac{1}{n+1}$ for all $k=0,1,\ldots,n$, the expression $\dfrac{1}{(n+1)^k}\displaystyle\sum_{\ell=0}^{k}\binom{k}{\ell}(2p-n)^{\lfloor \ell/2\rfloor}\gamma^{\ell \pmod 2}$ is equal to $\langle\tau_k\rangle$ (cf. [8]). As a result, the following limit is known:

$$\lim_{k\to\infty}\langle\tau_k\rangle = 0. \tag{3.7}$$

Observing that $t^+_{\underline{i}}(k)+t^-_{\underline{i}}(k)=\mathbb{P}\left(\tau_k=\pm\mathbf{e}_{\underline{i}}\right)$, the distribution of τ_k can be expressed.

Theorem 3.3. *Let $\mathbf{e}_{\underline{i}}$ be an arbitrary blade in $\mathcal{C}\ell_{p,q}$, and let k be an arbitrary positive integer. Then,*

$$\begin{aligned}\mathbb{P}\left(\tau_k=\mathbf{e}_{\underline{i}}\right) &= \frac{1}{2(n+1)^k}\sum_{\substack{\ell_0+\cdots+\ell_n=k\\ \ell_j \text{ odd if } 1\le j\in\underline{i},\ell_j \text{ even if } 1\le j\notin\underline{i}}}\binom{k}{\ell_0,\ldots,\ell_n} \\ &\quad + \frac{1}{2}\sum_{j_0=0}^{n}\cdots\sum_{j_{k-1}=0}^{n}\left(g_{j_0}(T^kx)-\frac{\varpi(\underline{i},\{j_0\})}{n+1}\right) \\ &\quad \times\left[\prod_{\ell=1}^{k-1}\left(g_{j_\ell}(T^{k-\ell}x)-\frac{1}{n+1}\varpi\left(\left(\underline{i}\underset{1\le m\le \ell-1}{\triangle}\{j_m\}\right),\{j_\ell\}\right)\right)\right].,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\mathbb{P}\left(\tau_k=-\mathbf{e}_{\underline{i}}\right) &= \frac{1}{2(n+1)^k}\sum_{\substack{\ell_0+\cdots+\ell_n=k\\ \ell_j \text{ odd if } 1\le j\in\underline{i},\ell_j \text{ even if } 1\le j\notin\underline{i}}}\binom{k}{\ell_0,\ldots,\ell_n} \\ &\quad - \frac{1}{2}\sum_{j_0=0}^{n}\cdots\sum_{j_{k-1}=0}^{n}\left(g_{j_0}(T^kx)-\frac{\varpi(\underline{i},\{j_0\})}{n+1}\right) \\ &\quad \times\left[\prod_{\ell=1}^{k-1}\left(g_{j_\ell}(T^{k-\ell}x)-\frac{1}{n+1}\varpi\left(\left(\underline{i}\underset{1\le m\le \ell-1}{\triangle}\{j_m\}\right),\{j_\ell\}\right)\right)\right]..\end{aligned} \tag{3.9}$$

Proof. The result follows from the following observation:

$$\mathbb{P}\left(\tau_k = \mathbf{e}_{\underline{i}}\right) = t^+_{\underline{i}}(k) = \frac{1}{2}\left[\left(t^+_{\underline{i}}(k) + t^-_{\underline{i}}(k)\right) + \left(t^+_{\underline{i}}(k) - t^-_{\underline{i}}(k)\right)\right]$$
$$= \frac{1}{2}\left[\mathbb{P}(\tau_k = \pm\mathbf{e}_{\underline{i}}) + \left\langle\langle\tau_k\rangle, \mathbf{e}_{\underline{i}}\right\rangle\right] \quad (3.10)$$

Similarly,

$$\mathbb{P}\left(\tau_k = -\mathbf{e}_{\underline{i}}\right) = t^-_{\underline{i}}(k) = \frac{1}{2}\left[\left(t^+_{\underline{i}}(k) + t^-_{\underline{i}}(k)\right) - \left(t^+_{\underline{i}}(k) - t^-_{\underline{i}}(k)\right)\right]. \quad (3.11)$$

□

Theorem 3.4. *If $\langle\tau_k\rangle \to 0$ as $k \to \infty$, then*

$$\tau_k \xrightarrow{\mathcal{D}} \mathcal{U}\left(\{\pm\mathbf{e}_{\underline{i}}\}\right). \quad (3.12)$$

Proof. If $\langle\tau_k\rangle \to 0$, then sufficiently large values of k give

$$\mathbb{P}\left(\tau_k = \mathbf{e}_{\underline{i}}\right) = \frac{1}{2(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k}{\ell_0, \ldots, \ell_n} + o(\varepsilon). \quad (3.13)$$

Moreover,

$$\mathbb{P}\left(\tau_k = -\mathbf{e}_{\underline{i}}\right) = \frac{1}{2(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\leq j\in\underline{i}, \ell_j \text{ even if } 1\leq j\notin\underline{i}}} \binom{k}{\ell_0, \ldots, \ell_n} + o(\varepsilon). \quad (3.14)$$

Turning now to the distribution of τ_k, recall Lemma 2.2. Passing to binary representations of subsets $\underline{i}$, each blade $\mathbf{e}_{\underline{i}} \in \mathcal{C}\ell_{p,q}$ is uniquely associated with a vertex of the n-dimensional hypercube. By identifying each pair $\pm\mathbf{e}_{\underline{i}}$, the walk (τ_k) induces a walk on the n-dimensional hypercube. The probability distribution of the k^{th} step of the associated hypercube random walk is determined by (2.11). Moreover, the limiting distribution of this walk is known to be uniform [1].

It then follows that

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle\tau_k, \mathbf{e}_{\underline{i}}\right\rangle = 1\right) = \lim_{k\to\infty} \mathbb{P}\left(\left\langle\tau_k, \mathbf{e}_{\underline{i}}\right\rangle = -1\right) = \frac{1}{2^{n+1}}, \quad (3.15)$$

and

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle\tau_k, \mathbf{e}_{\underline{i}}\right\rangle = 0\right) = \frac{2^n - 1}{2^n}. \quad (3.16)$$

□

Considering now the walk (ς_k),

$$
\begin{aligned}
\langle \varsigma_k \rangle = \prod_{i=1}^{k} \left(\sum_{j=1}^{n} \left(2 f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j \right) \\
= 2^k \sum_{j_1,\ldots,j_k \in [n]} \pm f_{j_1}(Tx) f_{j_2}(T^2 x) \cdots f_{j_k}(T^k x) \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_k} \\
+ \sum_{\ell=1}^{k-1} \frac{(2p-n)^{\lfloor (k-\ell)/2 \rfloor}}{n^{k-\ell}} \gamma^{(k-\ell) \pmod 2} \\
\times 2^\ell \sum_{\substack{j_1,\ldots,j_\ell \in [n] \\ h_1 \neq \cdots \neq h_\ell \in [k]}} \pm f_{j_1}(T^{h_1} x) \cdots f_{j_\ell}(T^{h_\ell} x) \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_\ell} \\
+ \frac{(-1)^k}{n^k} (2p-n)^{\lfloor k/2 \rfloor} \gamma^{k \pmod 2}.
\end{aligned} \tag{3.17}
$$

Lemma 3.5. *As $k \to \infty$,*

$$
\frac{1}{n^k} (2p-n)^{\lfloor k/2 \rfloor} \gamma^{k \pmod 2} \to 0. \tag{3.18}
$$

Proof. Note that $|2p-n| \leq n$ and $\lfloor k/2 \rfloor \leq k/2$ imply

$$
\left\| \left(\frac{(2p-n)^{1/2}}{n} \right)^k \right\| \leq \frac{1}{n^{k/2}}. \tag{3.19}
$$

Hence, for all $k > 0$,

$$
\left\| \frac{(2p-n)^{\lfloor k/2 \rfloor}}{n^k} \gamma^{k \pmod 2} \right\| \leq n \frac{1}{n^{k/2}} = \frac{1}{n^{(k-2)/2}}. \tag{3.20}
$$

□

Remark 3.6. In the time-homogeneous case given by $f_k(x) \equiv \frac{1}{n}$ for each $k = 0, \ldots, n$, the expression $\dfrac{1}{n^k}(2p-n)^{\lfloor k/2 \rfloor} \gamma^{k \pmod 2}$ represents $\langle \varsigma_k \rangle$ (cf. [8]).

Like the random walk (τ_k), $\lim\limits_{k \to \infty} \langle \varsigma_k \rangle$, if it exists, is given by

$$
\begin{aligned}
\lim_{k \to \infty} \langle \varsigma_k \rangle = \lim_{k \to \infty} 2^k \sum_{j_1,\ldots,j_k \in [n]} \pm f_{j_1}(Tx) f_{j_2}(T^2 x) \cdots f_{j_k}(T^k x) \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_k} \\
+ \sum_{\ell=1}^{k-1} \frac{(2p-n)^{\lfloor (k-\ell)/2 \rfloor}}{n^{k-\ell}} \gamma^{(k-\ell) \pmod 2} \\
\times 2^\ell \sum_{\substack{j_1,\ldots,j_\ell \in [n] \\ h_1 \neq \cdots \neq h_\ell \in [k]}} \pm f_{j_1}(T^{h_1} x) \cdots f_{j_\ell}(T^{h_\ell} x) \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_\ell}
\end{aligned} \tag{3.21}
$$

Unlike the walk (τ_k), the walk (ς_k) alternates between blades of even and odd degree. Hence, for each $k \geq 0$,

$$\langle \varsigma_k \rangle = \sum_{\substack{\underline{i} \in 2^{[n]} \\ |\underline{i}| \cong k \pmod 2}} \alpha_{\underline{i}}\, \mathbf{e}_{\underline{i}}. \tag{3.22}$$

An immediate consequence of this behavior is the following theorem.

Theorem 3.7. *If $\exists \lambda \in \mathcal{C}\ell_{p,q}$ such that $\lim_{k\to\infty} \langle \varsigma_k \rangle = \lambda$, then $\lambda = 0$.*

Like τ_k, with probability 1, $\varsigma_k = \pm \mathbf{e}_{\underline{i}}$ for some $\underline{i} \in 2^{[n]}$. Hence, for all $k > 0$,

$$||\varsigma_k|| = 1. \tag{3.23}$$

Theorem 3.8. *Let $\mathbf{e}_{\underline{i}}$ be an arbitrary blade in $\mathcal{C}\ell_{p,q}$, and let k be an arbitrary positive integer. Then,*

$$\begin{aligned}
\mathbb{P}\left(\varsigma_k = \mathbf{e}_{\underline{i}}\right) = \frac{1}{2n^k} &\sum_{\substack{\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd if } j \in \underline{i}, \ell_j \text{ even if } j \notin \underline{i}}} \binom{k}{\ell_1, \ldots, \ell_n} \\
&+ \frac{1}{2} \sum_{j_0=1}^{n} \cdots \sum_{j_{k-1}=1}^{n} \left(f_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n} \right) \\
&\times \left[\prod_{\ell=1}^{k-1} \left(f_{j_\ell}(T^{k-\ell}x) - \frac{1}{n} \varpi\left(\left(\underline{i} \mathop{\triangle}_{1 \leq m \leq \ell-1} \{j_m\} \right), \{j_\ell\} \right) \right) \right],
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
\mathbb{P}\left(\varsigma_k = -\mathbf{e}_{\underline{i}}\right) = \frac{1}{2n^k} &\sum_{\substack{\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd if } j \in \underline{i}, \ell_j \text{ even if } j \notin \underline{i}}} \binom{k}{\ell_1, \ldots, \ell_n} \\
&- \frac{1}{2} \sum_{j_0=1}^{n} \cdots \sum_{j_{k-1}=1}^{n} \left(f_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n} \right) \\
&\times \left[\prod_{\ell=1}^{k-1} \left(f_{j_\ell}(T^{k-\ell}x) - \frac{1}{n} \varpi\left(\left(\underline{i} \mathop{\triangle}_{1 \leq m \leq \ell-1} \{j_m\} \right), \{j_\ell\} \right) \right) \right].
\end{aligned} \tag{3.25}$$

Proof. Proof is similar to that of Theorem 3.3. □

Theorem 3.9. *If $\langle \varsigma_k \rangle \to 0$ as $k \to \infty$, then*

$$\varsigma_{2k} \xrightarrow{\mathcal{D}} \mathcal{U}\left(\{\pm \mathbf{e}_{\underline{i}} : |\underline{i}| \equiv 0 \pmod 2\}\right), \tag{3.26}$$

$$\varsigma_{2k-1} \xrightarrow{\mathcal{D}} \mathcal{U}\left(\{\pm \mathbf{e}_{\underline{i}} : |\underline{i}| \equiv 1 \pmod 2\}\right). \tag{3.27}$$

Proof. If $\langle \varsigma_k \rangle \to 0$, then sufficiently large values of k give the following when $k - |\underline{i}| \equiv 0 \pmod 2$.

$$\mathbb{P}\left(\varsigma_k = \mathbf{e}_{\underline{i}}\right) = \frac{1}{2n^k} \sum_{\substack{\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd if } j\in\underline{i}, \ell_j \text{ even if } j\notin\underline{i}}} \binom{k}{\ell_1,\ldots,\ell_n} + o(\varepsilon). \tag{3.28}$$

The even and odd subwalks then satisfy

$$\mathbb{P}\left(\varsigma_{2k} = \mathbf{e}_{\underline{i}} \middle| |\underline{i}| \equiv 0 \pmod 2\right) = \frac{1}{2n^k} \sum_{\substack{\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd if } j\in\underline{i}, \ell_j \text{ even if } j\notin\underline{i}}} \binom{k}{\ell_1,\ldots,\ell_n} + o(\varepsilon), \tag{3.29}$$

$$\mathbb{P}\left(\varsigma_{2k-1} = \mathbf{e}_{\underline{i}} \middle| |\underline{i}| \equiv 1 \pmod 2\right) = \frac{1}{2n^k} \sum_{\substack{\ell_1+\cdots+\ell_n=k \\ \ell_j \text{ odd if } j\in\underline{i}, \ell_j \text{ even if } j\notin\underline{i}}} \binom{k}{\ell_1,\ldots,\ell_n} + o(\varepsilon). \tag{3.30}$$

Each is proportional to the distribution of the random walk on the $(n-1)$-dimensional hypercube. When $|\underline{i}| \equiv 0 \pmod 2$,

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle = 1\right) = \lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle = -1\right) = \frac{1}{2^n}, \tag{3.31}$$

and

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle = 0\right) = \frac{2^{n-1}-1}{2^{n-1}}. \tag{3.32}$$

Further, when $|\underline{i}| \equiv 1 \pmod 2$,

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k-1}, \mathbf{e}_{\underline{i}} \right\rangle = 1\right) = \lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k-1}, \mathbf{e}_{\underline{i}} \right\rangle = -1\right) = \frac{1}{2^n}, \tag{3.33}$$

and

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k-1}, \mathbf{e}_{\underline{i}} \right\rangle = 0\right) = \frac{2^{n-1}-1}{2^{n-1}}. \tag{3.34}$$

□

3.1 Conditions for convergence

Conditions on the functions $\{g_j(x)\}$ such that $\langle \tau_k \rangle \to 0$ as $k \to \infty$ will now be discussed. The time-homogeneous case is considered first by fixing the transition probability $g_j(x)$ for $0 \le j \le n$.

Theorem 3.10. *Let α be a constant satisfying $0 \le \alpha \le \dfrac{1}{n+1}$. Defining $g_j(x) \equiv \alpha$ for $0 \le j \le n$, the walk $(\tau_k)_{k\ge 0}$ defined by* (2.10) *is time-homogeneous. Then $\langle \tau_k \rangle \to 0$ as $k \to \infty$ if $(\tau_k)_{k\ge 0}$ is defined on any Clifford algebra of signature other than* $(1,0)$. *The walk (τ_k) defined on $\mathcal{C}\ell_{1,0}$ converges if and only if $0 < \alpha < \dfrac{1}{2}$.*

Proof. Given α and random walk (τ_k) as described in the hypotheses of the theorem and applying Corollary 2.15,

$$\begin{aligned}
\langle\tau_k\rangle &= \prod_{i=1}^{k}\sum_{j=0}^{n}\left(2g_j(T^i x)-\frac{1}{n+1}\right)\mathbf{e}_j \\
&= \prod_{i=1}^{k}\sum_{j=0}^{n}\left(2\alpha-\frac{1}{n+1}\right)\mathbf{e}_j \\
&= \left(\sum_{j=0}^{n}\left(2\alpha-\frac{1}{n+1}\right)\mathbf{e}_j\right)^k \\
&= \left(2\alpha-\frac{1}{n+1}\right)^k \sum_{\substack{0\le\ell\le k\\ \ell \text{ even}}}\binom{k}{\ell}(2p-n)^{\ell/2} \\
&\quad + \left(2\alpha-\frac{1}{n+1}\right)^k \sum_{\substack{1\le\ell\le k\\ \ell \text{ odd}}}\binom{k}{\ell}(2p-n)^{(\ell-1)/2}\gamma. \qquad (3.35)
\end{aligned}$$

Let $P_k(z) = \sum_{\substack{0\le j\le k\\ j \text{ even}}}\binom{k}{j}z^{j/2}$, and let $Q_k(z) = \sum_{\substack{1\le j\le k\\ j \text{ odd}}}\binom{k}{j}z^{\frac{j-1}{2}}$ so that $P_k(z^2)+zQ_k(z^2) = (1+z)^k$. Putting $z^2 = 2p-n$, it becomes apparent that $\langle\tau_k\rangle \to 0$ if and only if $\left|(1+z)\left(2\alpha-\frac{1}{n+1}\right)\right| < 1$. Observe that for any choice of $\alpha \in \left[0, \frac{1}{n+1}\right]$, the following inequality holds: $\left|2\alpha-\frac{1}{n+1}\right| \le \frac{1}{n+1}$, so that convergence is guaranteed for all signatures except $(1,0)$ via

$$\left|(1+z)\left(2\alpha-\frac{1}{n+1}\right)\right| \le \left|\frac{1+z}{n+1}\right| = \frac{1+\sqrt{|p-q|}}{1+p+q} < 1. \qquad (3.36)$$

In signature $(1,0)$, convergence is guaranteed by the observation

$$\left|(1+z)\left(2\alpha-\frac{1}{2}\right)\right| = 2\left|2\alpha-\frac{1}{2}\right| \qquad (3.37)$$

with the observation that

$$\left|2\alpha-\frac{1}{2}\right| < \frac{1}{2} \Leftrightarrow 0<\alpha<\frac{1}{2}. \qquad (3.38)$$

$\square$

Before turning to the dynamic case, an auxiliary result is established.

Lemma 3.11. *Given $u,v\in \mathcal{C}\ell_{p,q}$, the inner product norm satisfies the following inequality:*

$$\|uv\| \le 2^{\frac{n}{2}}\|u\|\cdot\|v\|. \qquad (3.39)$$

Proof. By definition of the inner product norm and application of Schwartz' Inequality,

$$\|uv\|^2 = \sum_{\underline{k}\in 2^{[n]}} (uv)_{\underline{k}}^2 \le \sum_{\underline{k}\in 2^{[n]}} \left(\sum_{\underline{i}\triangle\underline{j}=\underline{k}} |u_{\underline{i}}v_{\underline{j}}|\right)^2 = \sum_{\underline{k}\in 2^{[n]}} \left(\sum_{\underline{j}\in 2^{[n]}} |u_{\underline{j}}v_{\underline{j}\triangle\underline{k}}|\right)^2$$
$$\le \sum_{\underline{k}\in 2^{[n]}} (\|u\|\cdot\|v\|)^2 \le 2^n\|u\|^2\|v\|^2. \quad (3.40)$$

□

Theorem 3.12. *Let $(\tau_k)_{k\ge 0}$ be the dynamic random walk defined by* (2.10)*. Then a sufficient condition for $\langle\tau_k\rangle \to 0$ as $k\to\infty$ is*

$$\left|\sum_{j=0}^{n} g_j(T^i x) - \frac{1}{2(n+1)}\right| = O\left(\frac{1}{2^{(n+3)/2}}\right),\ \forall i\ge 0.$$

Proof. Given the random walk (τ_k) as described in the hypotheses of the theorem,

$$||\langle\tau_k\rangle|| = \left\|\prod_{i=1}^{k}\sum_{j=0}^{n}\left(2g_j(T^i x) - \frac{1}{n+1}\right)\mathbf{e}_j\right\|$$
$$\le 2^{nk/2}\prod_{i=1}^{k}\left\|\sum_{j=0}^{n}\left(2g_j(T^i x) - \frac{1}{n+1}\right)\mathbf{e}_j\right\|. \quad (3.41)$$

Observe that

$$\left\|\sum_{j=0}^{n}\left(2g_j(T^i x) - \frac{1}{n+1}\right)\mathbf{e}_j\right\| = \left|\sum_{j=0}^{n}\left(2g_j(T^i x) - \frac{1}{n+1}\right)\right| = O\left(\frac{1}{2^{n/2+1}}\right). \quad (3.42)$$

Thus,

$$||\langle\tau_k\rangle|| = O\left(2^{nk/2}\prod_{i=1}^{k}\frac{1}{2^{n/2+1}}\right) = O\left(\frac{2^{nk/2}}{2^{nk/2+k}}\right) = O\left(\frac{1}{2^k}\right). \quad (3.43)$$

Hence, $\lim_{k\to\infty}\langle\tau_k\rangle = 0$. □

Similar conditions for convergence apply to the random walk $(\varsigma_k)_{k\ge 0}$. The time-homogeneous case is considered first.

Theorem 3.13. *Let α be a fixed constant satisfying $0\le\alpha\le\dfrac{1}{n}$. Defining $f_j(x)\equiv\alpha$ for $1\le j\le n$, the walk $(\varsigma_k)_{k\ge 0}$ defined by* (2.4) *is time-homogeneous. Then $\langle\varsigma_k\rangle\to 0$ as $k\to\infty$ if $(\varsigma_k)_{k\ge 0}$ is defined on any Clifford algebra of signature other than* $(1,0)$*. The walk (ς_k) defined on $\mathcal{C}\ell_{1,0}$ converges if and only if $0<\alpha<1$.*

Proof. Given α and random walk (ς_k) as described in the hypotheses of the theorem and applying Corollary 2.14,

$$\begin{aligned}\langle\varsigma_k\rangle = \prod_{i=1}^{k}\sum_{j=1}^{n}\left(2f_j(T^i x) - \frac{1}{n}\right)\mathbf{e}_j \\ = \prod_{i=1}^{k}\sum_{j=1}^{n}\left(2\alpha - \frac{1}{n}\right)\mathbf{e}_j \\ = \left(\sum_{j=1}^{n}\left(2\alpha - \frac{1}{n}\right)\mathbf{e}_j\right)^k \\ = \left(2\alpha - \frac{1}{n}\right)^k (2p-n)^{\lfloor k/2\rfloor}. \quad (3.44)\end{aligned}$$

It becomes apparent that $\langle\varsigma_k\rangle \to 0$ if and only if

$$\left|\sqrt{|2p-n|}\left(2\alpha - \frac{1}{n}\right)\right| < 1.$$

Observe that for any choice of $\alpha \in \left[0, \frac{1}{n}\right]$, the following inequality holds: $\left|2\alpha - \frac{1}{n}\right| \leq \frac{1}{n}$, so that convergence is guaranteed for all signatures except $(1,0)$ via

$$\left|\sqrt{|2p-n}\left(2\alpha - \frac{1}{n}\right)\right| \leq \left|\frac{\sqrt{|2p-n|}}{n}\right| = \frac{\sqrt{|p-q|}}{p+q} < 1. \quad (3.45)$$

In signature $(1,0)$, convergence is guaranteed by choosing $0 < \alpha < 1$. With this assumption,

$$\left|\sqrt{|2p-n|}\left(2\alpha - \frac{1}{n}\right)\right| = |2\alpha - 1| < 1. \quad (3.46)$$

□

Theorem 3.14. *Let $(\varsigma_k)_{k\geq 0}$ be the dynamic random walk defined by* (2.4)*. Then a sufficient condition for $\langle\varsigma_k\rangle \to 0$ as $k \to \infty$ is*

$$\left|\sum_{j=0}^{n} f_j(T^i x) - \frac{1}{2n}\right| = O\left(\frac{1}{2^{(n+3)/2}}\right), \forall i \geq 0.$$

Proof. Given the random walk (ς_k) as described in the hypotheses of the theorem,

$$||\langle\varsigma_k\rangle|| = \left\| \prod_{i=1}^{k} \sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j \right\|$$
$$\leq 2^{nk/2} \prod_{i=1}^{k} \left\| \sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j \right\| . \quad (3.47)$$

Observe that

$$\left\| \sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \mathbf{e}_j \right\| = \left| \sum_{j=1}^{n} \left(2f_j(T^i x) - \frac{1}{n} \right) \right| = O\left(\frac{1}{2^{n/2+1}} \right) . \quad (3.48)$$

Thus,

$$||\langle\varsigma_k\rangle|| = O\left(2^{nk/2} \prod_{i=1}^{k} \frac{1}{2^{n/2+1}} \right) = O\left(\frac{2^{nk/2}}{2^{nk/2+k}} \right) = O\left(\frac{1}{2^k} \right) . \quad (3.49)$$

It follows that $\lim_{k\to\infty} \langle\varsigma_k\rangle = 0$. □

3.2 Induced Additive Walks

Given multiplicative walks (ς_k) and (τ_k), define the additive walks (Ξ_N) and (Υ_N) by

$$\Xi_N = \sum_{k=1}^{N} \varsigma_k, \quad (3.50)$$
$$\Upsilon_N = \sum_{k=1}^{N} \tau_k. \quad (3.51)$$

Moreover, define the *even* and *odd* additive walks $({\Xi_N}^+)$ and $({\Xi_N}^-)$ by

$${\Xi_N}^+ = \sum_{k=1}^{N} \varsigma_{2k}, \quad (3.52)$$
$${\Xi_N}^- = \sum_{k=1}^{N} \varsigma_{2k-1}. \quad (3.53)$$
$$(3.54)$$

Recalling $\varsigma_k = M_1 M_2 \cdots M_k$ and $\tau_k = L_1 L_2 \cdots L_k$,

$$\langle M_1 \cdots M_\ell \rangle = \langle M_1 \cdots M_k \rangle \langle M_{k+1} \cdots M_\ell \rangle . \quad (3.55)$$

Similarly,

$$\langle L_1 \cdots L_\ell \rangle = \langle L_1 \cdots L_k \rangle \langle L_{k+1} \cdots L_\ell \rangle . \tag{3.56}$$

Note that for $N > 0$, values of coefficients in Ξ_N are bounded according to $0 \leq |\langle \Xi_N, \mathbf{e}_{\underline{i}} \rangle| \leq \frac{N-|\underline{i}|+1}{2}$. Note also that for $N > 0$, values of coefficients in Υ_N are bounded according to $0 \leq |\langle \Upsilon_N, \mathbf{e}_{\underline{i}} \rangle| \leq N - |\underline{i}| + 1$.

The goal is to prove a law of large numbers and a central limit theorem for the walks $(\Xi_N)_{N>0}$ and $(\Upsilon_N)_{N>0}$.

Let $\underline{i} \in 2^{[n]}$ be arbitrary. Note that for each $N > 0$, linearity of expectation gives

$$\begin{aligned}\mathbb{E}\left(\langle \Upsilon_N, \mathbf{e}_{\underline{i}} \rangle\right) &= \sum_{k=1}^{N} \sum_{j_0=0}^{n} \cdots \sum_{j_{k-1}=0}^{n} \left(g_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n+1} \right) \\ &\times \left[\prod_{\ell=1}^{k-1} \left(g_{j_\ell}(T^{k-\ell} x) - \frac{1}{n+1} \varpi\left(\left(\underline{i} \underset{1\leq m \leq \ell-1}{\triangle} \{j_m\} \right), \{j_\ell\} \right) \right) \right] .\end{aligned} \tag{3.57}$$

Proposition 3.15. *If $\langle \varsigma_k \rangle \to 0$ as $k \to \infty$, then the following limit exists:*

$$\tilde{\Xi} = \lim_{N \to \infty} \langle \Xi_N \rangle . \tag{3.58}$$

Similarly, if $\langle \tau_k \rangle \to 0$ as $k \to \infty$, then the following limit exists:

$$\tilde{\Upsilon} = \lim_{N \to \infty} \langle \Upsilon_N \rangle . \tag{3.59}$$

Proof. If $\langle \tau_k \rangle \to 0$ as $k \to \infty$, then $\tau_k \xrightarrow{\mathcal{D}} \mathcal{U}(\{\pm \mathbf{e}_{\underline{i}}\})$ implies that given arbitrary $\varepsilon > 0$, there exists N_ε such that for any $N > N_\varepsilon$, $||\langle \Upsilon_N - \Upsilon_{N_\varepsilon} \rangle|| < \varepsilon$. Hence,

$$\langle \Upsilon_N \rangle = \langle \Upsilon_N - \Upsilon_{N_\varepsilon} \rangle + \langle \Upsilon_{N_\varepsilon} \rangle \tag{3.60}$$

guarantees convergence.

The proof of convergence of Ξ_N is done analogously considering the subwalks ${\Xi_N}^+$ and ${\Xi_N}^-$. □

The next result shows that the limiting expectations $\tilde{\Upsilon}$ and $\tilde{\Xi}$ depend only on the expected values of τ_1 and ς_1, respectively.

Theorem 3.16. *If $\langle \tau_k \rangle \to 0$ as $k \to \infty$, then*

$$\tilde{\Upsilon} = \sum_{\ell=1}^{\infty} \left(\sum_{j=0}^{n} \left(2 g_j(Tx) - \frac{1}{n+1} \right) \mathbf{e}_j \right)^{\ell} . \tag{3.61}$$

Similarly, if $\langle \varsigma_k \rangle \to 0$ as $k \to \infty$, then

$$\tilde{\Xi} = \sum_{\ell=1}^{\infty} \left(\sum_{j=1}^{n} \left(2 f_j(Tx) - \frac{1}{n} \right) \mathbf{e}_j \right)^{\ell} . \tag{3.62}$$

Proof. Given $\Upsilon_N = Q_1 + Q_1Q_2 + \cdots + (Q_1\cdots Q_N)$, write $\Upsilon'_N = Q_2 + Q_2Q_3 + \cdots + (Q_2\cdots Q_N)$ so that

$$\langle \Upsilon_N \rangle = \langle \Upsilon_1 \rangle \left(1 + \langle \Upsilon'_N \rangle\right). \tag{3.63}$$

Given that the limit $\tilde{\Upsilon}$ exists, passing to the limit then gives

$$\tilde{\Upsilon} = \langle \Upsilon_1 \rangle (1 + \tilde{\Upsilon}), \tag{3.64}$$

so that

$$\tilde{\Upsilon} = \frac{\langle \Upsilon_1 \rangle}{(1 - \langle \Upsilon_1 \rangle)}. \tag{3.65}$$

Then,

$$\frac{\langle \Upsilon_1 \rangle}{(1 - \langle \Upsilon_1 \rangle)} = \langle \Upsilon_1 \rangle (1 + \langle \Upsilon_1 \rangle + \langle \Upsilon_1 \rangle^2 + \cdots), \tag{3.66}$$

which implies the result.

An analogous argument for $\tilde{\Xi}$ completes the proof. □

Remark 3.17. In the time-homogeneous case, the limiting expectations are found to be paravector-valued.

Lemma 3.18. *For fixed* $\underline{i} \in 2^{[n]}$*, the variance of* $\left\langle \Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle$ *is given by*

$$\begin{aligned} \sigma_\Upsilon(N, \underline{i})^2 &= \operatorname{var}\left(\left\langle \Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle\right) \\ &= \mathbb{E}\left(\left\langle \Upsilon_N, \mathbf{e}_{\underline{i}} \right\rangle^2\right) - \mathbb{E}\left(\left\langle \Upsilon_N, \mathbf{e}_{\underline{i}} \right\rangle\right)\left(1 + 2\left\langle \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle\right) + \left\langle \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle^2 + \left\langle \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle, \end{aligned} \tag{3.67}$$

where

$$\begin{aligned} \mathbb{E}\left(\left\langle \Upsilon_N, \mathbf{e}_{\underline{i}} \right\rangle\right) = \sum_{k=1}^{N} \sum_{j_0=0}^{n} \cdots \sum_{j_{k-1}=0}^{n} &\left(g_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n+1} \right) \\ \times &\left[\prod_{\ell=1}^{k-1} \left(g_{j_\ell}(T^{k-\ell}x) - \frac{1}{n+1} \varpi\left(\left(\underline{i} \mathop{\triangle}_{1 \le m \le \ell-1} \{j_m\}\right), \{j_\ell\}\right)\right)\right] \end{aligned} \tag{3.68}$$

and

$$\begin{aligned} \mathbb{E}\left(\left\langle \Upsilon_N, \mathbf{e}_{\underline{i}} \right\rangle^2\right) = \sum_{k=1}^{N} \frac{1}{(n+1)^k} &\sum_{\substack{\ell_0 + \cdots + \ell_n = k \\ \ell_j \text{ odd if } 1 \le j \in \underline{i}, \ell_j \text{ even if } 1 \le j \notin \underline{i}}} \binom{k}{\ell_0, \ldots, \ell_n} \\ &+ 2 \sum_{1 \le k < j \le N} \mathbb{E}\left(\left\langle \tau_k, \mathbf{e}_{\underline{i}} \right\rangle \left\langle \tau_j, \mathbf{e}_{\underline{i}} \right\rangle\right). \end{aligned} \tag{3.69}$$

Proof. By Theorem 3.3,

$$\mathbb{E}\left(\langle \tau_k, \mathbf{e}_{\underline{i}}\rangle\right) = \sum_{j_0=0}^{n} \cdots \sum_{j_{k-1}=0}^{n} \left(g_{j_0}(T^k x) - \frac{\varpi(\underline{i}, \{j_0\})}{n+1} \right) \times \left[\prod_{\ell=1}^{k-1} \left(g_{j_\ell}(T^{k-\ell}x) - \frac{1}{n+1} \varpi\left(\left(\underline{i} \mathop{\triangle}_{1\le m\le \ell-1} \{j_m\} \right), \{j_\ell\} \right) \right) \right]. \quad (3.70)$$

Summing over k then gives $\mathbb{E}\left(\langle \Upsilon_N, \mathbf{e}_{\underline{i}}\rangle\right)$.

On the other hand,

$$\begin{aligned}
\mathbb{E}\left(\langle \Upsilon_N, \mathbf{e}_{\underline{i}}\rangle^2\right) &= \mathbb{E}\left(\left(\sum_{k=1}^{N} \langle \tau_k, \mathbf{e}_{\underline{i}}\rangle \right)^2 \right) \\
&= \mathbb{E}\left(\sum_{k=1}^{N} \langle \tau_k, \mathbf{e}_{\underline{i}}\rangle^2 + 2 \sum_{1\le k<j\le N} \langle \tau_k, \mathbf{e}_{\underline{i}}\rangle \langle \tau_j, \mathbf{e}_{\underline{i}}\rangle \right) \\
&= \sum_{k=1}^{N} \frac{1}{(n+1)^k} \sum_{\substack{\ell_0+\cdots+\ell_n=k \\ \ell_j \text{ odd if } 1\le j\in\underline{i},\, \ell_j \text{ even if } 1\le j\notin\underline{i}}} \binom{k}{\ell_0,\ldots,\ell_n} \\
&\quad + 2 \sum_{1\le k<j\le N} \mathbb{E}\left(\langle \tau_k, \mathbf{e}_{\underline{i}}\rangle \langle \tau_j, \mathbf{e}_{\underline{i}}\rangle\right). \qquad (3.71)
\end{aligned}$$

□

3.3 Central Limit Theorems

It has now been established that under appropriate conditions, such as those indicated in Theorems 3.13 and 3.14, $\tilde{\Xi}$ exists, and for each $N > 0$,

$$\operatorname{var}\left(\frac{\left\langle \Xi_N - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Xi(N, \underline{i})} \right) = 1. \qquad (3.72)$$

Moreover, as $N \to \infty$,

$$\mathbb{E}\left(\frac{\left\langle \Xi_N - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Xi(N, \underline{i})} \right) \to 0. \qquad (3.73)$$

Similarly, under conditions such as those in Theorems 3.10 and 3.12, $\tilde{\Upsilon}$ exists, and for each $N > 0$,

$$\operatorname{var}\left(\frac{\left\langle \Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Upsilon(N, \underline{i})} \right) = 1. \qquad (3.74)$$

As $N \to \infty$,

$$\mathbb{E}\left(\frac{\left\langle \Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Upsilon(N, \underline{i})}\right) \to 0. \tag{3.75}$$

Characterizing the limiting distributions of these random variables is all that remains.

Theorem 3.19. *If $\langle \varsigma_k \rangle \to 0$ as $k \to \infty$, then*

$$\frac{\left\langle \Xi_N - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Xi(N, \underline{i})} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.76}$$

Similarly, if $\langle \tau_k \rangle \to 0$ as $k \to \infty$, then

$$\frac{\left\langle \Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Upsilon(N, \underline{i})} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.77}$$

Proof. Define i.i.d. collections of Bernoulli random variables X_i and Y_i such that for each $i \in \mathbb{N}$, X_i takes values in $\{0, 1\}$ such that

$$\mathbb{P}(X_i = 1) = \frac{1 + \sqrt{1 - \dfrac{2^n - 1}{2^{n-1}}}}{2}, \tag{3.78}$$

$$\mathbb{P}(X_i = 0) = \frac{1 - \sqrt{1 - \dfrac{2^n - 1}{2^{n-1}}}}{2}. \tag{3.79}$$

For each $i \in \mathbb{N}$, Y_i takes values in $\{-1, 0\}$ such that

$$\mathbb{P}(Y_i = -1) = \frac{1 + \sqrt{1 - \dfrac{2^n - 1}{2^{n-1}}}}{2}, \tag{3.80}$$

$$\mathbb{P}(Y_i = 0) = \frac{1 - \sqrt{1 - \dfrac{2^n - 1}{2^{n-1}}}}{2}. \tag{3.81}$$

Note that for each positive integer i,

$$\operatorname{var}(X_i) = \operatorname{var}(Y_i) = \frac{1 + \sqrt{1 - \dfrac{2^n - 1}{2^{n-1}}}}{2}. \tag{3.82}$$

Define $S_N = \sum_{k=1}^N X_k$ and $T_N = \sum_{k=1}^N T_k$. By the central limit theorem,

$$\frac{\sqrt{2}S_N}{\sqrt{N\left(1+\sqrt{1-\dfrac{2^n-1}{2^{n-1}}}\right)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \tag{3.83}$$

$$\frac{\sqrt{2}T_N}{\sqrt{N\left(1+\sqrt{1-\dfrac{2^n-1}{2^{n-1}}}\right)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.84}$$

For each $i = 1, 2, \ldots$, $Z_i = X_i + Y_i$ is a random variable taking values in $\{-1, 0, 1\}$ such that

$$\mathbb{P}(Z_i = -1) = \frac{1}{2^n}, \tag{3.85}$$

$$\mathbb{P}(Z_i = 0) = \frac{2^{n-1}-1}{2^{n-1}}, \tag{3.86}$$

$$\mathbb{P}(Z_i = 1) = \frac{1}{2^n}. \tag{3.87}$$

For each positive integer i,

$$\operatorname{var}(X_i + Y_i) = \operatorname{var}(Z_i) = \frac{1}{2^{n-1}}. \tag{3.88}$$

Hence, defining $W_N = S_N + T_N$, the limiting distribution is the sum of two Gaussian random variables and

$$\frac{W_N}{\sqrt{N2^{n-1}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.89}$$

The limiting distribution function is then given by

$$\mathbb{P}\left(W \le x\right) = \lim_{N\to\infty} \frac{1}{\sqrt{N2^n\pi}} \int_{-\infty}^{x} \exp\left(-\frac{2^{n-1}y^2}{2N}\right)\, dy. \tag{3.90}$$

If $\langle \varsigma_k \rangle \to 0$ as $k \to \infty$, the limiting distribution of ς_{2k} is uniform on the positive and negative basis multivectors of even degree in $\mathcal{C}\ell_{p,q}$ by Theorem 3.9. It follows that for each $\underline{i} \in 2^{[n]}$, $\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle\right)_{k>0}$ is a sequence of random variables having values in $\{-1, 0, 1\}$ with limiting distribution

$$\lim_{k\to\infty} \mathbb{P}\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle = \beta\right) = \begin{cases} \dfrac{1}{2^n} & \text{if } \beta = 1 \\ 1 - \dfrac{1}{2^{n-1}} & \text{if } \beta = 0 \\ \dfrac{1}{2^n} & \text{if } \beta = -1. \end{cases} \tag{3.91}$$

In other words, for arbitrary $\underline{i} \in 2^{[n]}$, as $k \to \infty$

$$\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle \xrightarrow{\mathcal{D}} Z_i. \tag{3.92}$$

It also follows that

$$\lim_{k\to\infty} \operatorname{var}\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle\right) = \frac{1}{2^{n-1}}. \tag{3.93}$$

Convergence in distribution of the sequence $\left(\left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle\right)_{k>0}$ implies that for any $\varepsilon > 0$, there exists N_ε such that $N > N_\varepsilon$ implies

$$F_{N-N_\varepsilon}(x) - \varepsilon \leq \mathbb{P}\left(\sum_{k=N_\varepsilon}^{N} \left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle \leq x\right) \leq F_{N-N_\varepsilon}(x) + \varepsilon, \tag{3.94}$$

where writing $M = N - N_\varepsilon$ yields

$$F_M(x) = \sum_{\kappa=-M}^{\lfloor x \rfloor} \psi_M(\kappa). \tag{3.95}$$

Here $\psi_M(x)$ is the M^{th} mass function defined by

$$\psi_M(x) = \left(\frac{1}{2^n}\right)^{|x|} \sum_{k=0}^{\lfloor \frac{M-|x|}{2} \rfloor} \binom{M}{k} \binom{M-k}{|x|+k} \left(\frac{1}{2^n}\right)^{2k} \left(\frac{2^{n-1}-1}{2^{n-1}}\right)^{M-(|x|+2k)}. \tag{3.96}$$

which has support $\{-M, \ldots, M\}$ and gives the probability that the sum of M random variables taking values in $\{-1, 0, 1\}$ with respective probabilities $\{\frac{1}{2^n}, \frac{2^{n-1}-1}{2^{n-1}}, \frac{1}{2^n}\}$ is equal to x.

Because $\psi_M(x)$ is associated with a sum of independent Bernoulli random variables each converging to a Gaussian random variable, the functions $\{\psi_M\}$ converge to the mass function of a Gaussian random variable as $M \to \infty$.

Fix $x \in \mathbb{R}$ and $N_\varepsilon > 0$. As $N \to \infty$, $|F_N(x) - F_{N-N_\varepsilon}(x)| \to 0$. Observing that $\sum_{k=1}^{N} \left\langle \varsigma_{2k}, \mathbf{e}_{\underline{i}} \right\rangle = \left\langle {\Xi_N}^+, \mathbf{e}_{\underline{i}} \right\rangle$ and replacing Ξ_{N_ε} by $\tilde{\Xi}$, one finds

$$\left| \mathbb{P}\left(\left\langle {\Xi_N}^+ - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle \leq x\right) - F_N(x) \right| \to 0. \tag{3.97}$$

Therefore,

$$\frac{\left\langle {\Xi_N}^+ - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Xi(N, \underline{i})} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.98}$$

Similarly,

$$\frac{\left\langle {\Xi_N}^- - \tilde{\Xi}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Xi(N, \underline{i})} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.99}$$

The characterization of the limiting distribution of $\dfrac{\Upsilon_N - \tilde{\Upsilon}}{\sigma_\Upsilon(N, \underline{i})}$ is established in an analogous manner. Define i.i.d. collections of Bernoulli random variables ${X_i}'$ and ${Y_i}'$ such that for each $i \in \mathbb{N}$, ${X_i}'$ takes values in $\{0, 1\}$, and ${Y_i}'$ takes values in $\{0, -1\}$ with

$$\mathbb{P}({X_i}' = 1) = \mathbb{P}({Y_i}' = -1) = \frac{1 + \sqrt{1 - \dfrac{2^{n+1} - 1}{2^n}}}{2}, \tag{3.100}$$

$$\mathbb{P}({X_i}' = 0) = \mathbb{P}({Y_i}' = 0) = \frac{1 - \sqrt{1 - \dfrac{2^{n+1} - 1}{2^n}}}{2}. \tag{3.101}$$

Then, for each $i = 1, 2, \ldots$, ${Z_i}' = {X_i}' + {Y_i}'$ is a random variable taking values in $\{-1, 0, 1\}$ such that

$$\mathbb{P}({Z_i}' = -1) = \frac{1}{2^{n+1}}, \tag{3.102}$$

$$\mathbb{P}({Z_i}' = 0) = \frac{2^n - 1}{2^n}, \tag{3.103}$$

$$\mathbb{P}({Z_i}' = 1) = \frac{1}{2^{n+1}}. \tag{3.104}$$

For each positive integer i,

$$\operatorname{var}({X_i}' + {Y_i}') = \operatorname{var}({Z_i}') = \frac{1}{2^n}. \tag{3.105}$$

Define ${S_N}' = \sum_{k=1}^N {X_k}'$ and ${T_N}' = \sum_{k=1}^N {T_k}'$. By the central limit theorem,

$$\frac{\sqrt{2}{S_N}'}{\sqrt{N\left(1 + \sqrt{1 - \dfrac{2^{n+1} - 1}{2^n}}\right)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \tag{3.106}$$

$$\frac{\sqrt{2}{T_N}'}{\sqrt{N\left(1 + \sqrt{1 - \dfrac{2^{n+1} - 1}{2^n}}\right)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1). \tag{3.107}$$

Hence, defining ${W_N}' = {S_N}' + {T_N}'$, the limiting distribution is the sum of two Gaussian random variables and

$$\frac{{W_N}'}{\sqrt{N 2^n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1). \tag{3.108}$$

The limiting distribution function is then given by

$$\mathbb{P}\left(W' \leq x\right) = \lim_{N \to \infty} \frac{1}{\sqrt{N 2^{n+1} \pi}} \int_{-\infty}^{x} \exp\left(-\frac{2^n y^2}{2N}\right) dy. \tag{3.109}$$

If $\langle\tau_k\rangle \to 0$ as $k \to \infty$, the limiting distribution of τ_k is uniform on the positive and negative basis multivectors of $\mathcal{C}\ell_{p,q}$ by Theorem 3.4. It follows that for each $\underline{i} \in 2^{[n]}$, $\left(\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle\right)_{k>0}$ is a sequence of random variables having values in $\{-1, 0, 1\}$ with limiting distribution

$$\lim_{k\to\infty} \mathbb{P}\left(\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle = \beta\right) = \begin{cases} 2^{-(n+1)} & \text{if } \beta = 1 \\ 1 - 2^{-n} & \text{if } \beta = 0 \\ 2^{-(n+1)} & \text{if } \beta = -1. \end{cases} \tag{3.110}$$

It also follows that

$$\lim_{k\to\infty} \operatorname{var}\left(\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle\right) = \frac{1}{2^n}. \tag{3.111}$$

For arbitrary $\underline{i} \in 2^{[n]}$, as $k \to \infty$

$$\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle \xrightarrow{\mathcal{D}} {Z_i}'. \tag{3.112}$$

Convergence in distribution of the sequence $\left(\langle\tau_k, \mathbf{e}_{\underline{i}}\rangle\right)_{k>0}$ implies that for any $\varepsilon > 0$, there exists N_ε such that $N > N_\varepsilon$ implies

$$G_{N-N_\varepsilon}(x) - \varepsilon \leq \mathbb{P}\left(\sum_{k=N_\varepsilon}^{N} \langle\tau_k, \mathbf{e}_{\underline{i}}\rangle \leq x\right) \leq G_{N-N_\varepsilon}(x) + \varepsilon, \tag{3.113}$$

where writing $M = N - N_\varepsilon$ yields

$$G_M(x) = \sum_{\kappa=-M}^{\lfloor x\rfloor} \phi_M(\kappa). \tag{3.114}$$

Here $\phi_M(x)$ is the M^{th} mass function defined by

$$\phi_M(x) = \left(\frac{1}{2^{n+1}}\right)^{|x|} \sum_{k=0}^{\lfloor\frac{M-|x|}{2}\rfloor} \binom{M}{k}\binom{M-k}{|x|+k} \frac{1}{(2^{n+1})^{2k}} \left(\frac{2^n-1}{2^n}\right)^{M-(|x|+2k)}. \tag{3.115}$$

which has support $\{-M, \ldots, M\}$ and gives the probability that the sum of M random variables taking values in $\{-1, 0, 1\}$ with respective probabilities $\{\frac{1}{2^{n+1}}, \frac{2^n-1}{2^n}, \frac{1}{2^{n+1}}\}$ is equal to x.

Because $\phi_M(x)$ is associated with a sum of independent Bernoulli random variables each converging to a Gaussian random variable, the functions $\{\phi_M\}$ converge to the mass function of a Gaussian random variable as $M \to \infty$.

Fix $x \in \mathbb{R}$ and $N_\varepsilon > 0$. As $N \to \infty$, $|G_N(x) - G_{N-N_\varepsilon}(x)| \to 0$. Observing that $\sum_{k=1}^{N} \langle\tau_k, \mathbf{e}_{\underline{i}}\rangle = \langle\Upsilon_N, \mathbf{e}_{\underline{i}}\rangle$ and replacing Υ_{N_ε} by $\tilde{\Upsilon}$, one finds

$$\left|\mathbb{P}\left(\left\langle\Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}}\right\rangle \leq x\right) - G_N(x)\right| \to 0. \tag{3.116}$$

Hence,

$$\frac{\left\langle \Upsilon_N - \tilde{\Upsilon}, \mathbf{e}_{\underline{i}} \right\rangle}{\sigma_\Upsilon(N, \underline{i})} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.117}$$

□

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